Substituting from the condition $nk^{2(n-1)} = 1$ we find

$$\frac{d^2T}{dk^2} = -\frac{d(n-1)}{k}\left(\frac{n-1}{a\log n}\right)^{1/2}.$$

Since this result is negative $(n > 1)$ a true maximum obtains.

Case (b). $1 > n > 0$.

Then $\overline{FP} < \overline{FQ}$. Proceeding as before we get

$$T' = -\int_{k^nd}^{kd}\left(2a\log\frac{d}{x}\right)^{-1/2}dx,$$

where T' symbolizes the interval of time consumed in going from Q to P. Consequently $nk^{2(n-1)} = 1$ and

$$\frac{d^2T'}{dk^2} = -\frac{d(1-n)}{k}\left(\frac{1-n}{-a\log n}\right)^{1/2}.$$

Since $n - 1$ is now negative, the second derivative is also negative, and hence the condition for a maximum is again fulfilled.

Remark: In making the final reductions in case (b) attention has to be paid to the fact that $\log n$ is negative as well as $n - 1$.

Also solved by A. H. Wilson, H. N. Carleton, H. Polish, J. A. Caparo, Elijah Swift, Paul Capron, and the Proposer.

NUMBER THEORY.

209. (March, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer.

Solution by Elijah Swift, University of Vermont.

In Carmichael's Diophantine Analysis, pp. 70, 71, the impossibility of the equation $x^3 + y^3 = 2^nz^3$ is proved. (The proof is valid whether y is positive or negative.)

We are to prove the impossibility of the equation $a^6 - b^6 = c^2$. We know that if $x^2 + y^2 = z^2$, then $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$. (Loc. cit., p. 10.) We assume, of course, that a, b, c are all prime to each other; so are x, y, z also. Two cases present themselves:

(I) $\quad a^3 = m^2 + n^2, \quad b^3 = m^2 - n^2, \quad c = 2mn.$

(II) $\quad a^3 = m^2 + n^2, \quad b^3 = 2mn, \quad c = m^2 - n^2.$

Case I. Of the two integers, m and n, one is odd and the other even. Also these numbers are prime to each other. Consequently $m + n$ and $m - n$ are prime to each other, and since their product is a cube, each of them must be a cube also. If we set them equal to α^3 and β^3 respectively

$$m + n = \alpha^3, \quad m - n = \beta^3, \quad a^3 = m^2 + n^2 = \frac{\alpha^6 + \beta^6}{2}, \quad \text{or} \quad \alpha^6 + \beta^6 = 2a^3,$$

the impossibility of which was stated in the first paragraph.

Case II. Assume m even; the proof will apply equally well to the other case, n even. Since $2mn$ is a cube, $2m$ and n must each of them be a cube. Setting them equal to $8\alpha^3$ and β^3, we have

$$m = 4\alpha^3, \qquad n = \beta^3, \qquad a^3 = 16\alpha^6 + \beta^6 \qquad \text{or} \qquad 2^4\alpha^6 = a^3 - \beta^6,$$

an equation which was proved to be impossible. Hence the equation $a^6 - b^6 = c^2$ cannot hold.

219. (June, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

Solution by Elijah Swift, University of Vermont.

If the number of sides of a polygon is n, the number of its diagonals is $n(n - 3)/2$, which by the conditions of the problem must be a perfect fourth power.